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Abstract

We consider a hydrogen atom in the background spacetimes generated by an infinitely thin cosmic string and by a point-like global monopole. In both cases, we find the solutions of the corresponding Dirac equations and we determine the energy levels of the atom. We investigate how the geometric and topological features of these spacetimes leads to shifts in the energy levels as compared with the flat Minkowski spacetime.

PACS numbers: 03.65.Ge, 03.65.Nk, 14.80.Hv

I. Introduction

The study of quantum systems in curved spacetimes goes back to the end of twenties and to the beginning of thirties of the last century [1], when the generalization of the Schrödinger and Dirac equations to curved spaces has been discussed, motivated by the idea of constructing a theory which combines quantum physics and general relativity.

Spinor fields and particles interacting with gravitational fields has been the subject of many investigations. Along this line of research we can mention those concerning the

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determination of the renormalized vacuum expectation value of the energy-momentum tensor and the problem of creation of particles in expanding Universes [2], and those connected with quantum mechanics in different background spacetimes [3] and, in particular, the ones which consider the hydrogen atom[4-8] in an arbitrary curved spacetime.

The study of the single-particle states which are exact solutions of the generalized Dirac equation in curved spacetimes constitutes an important element to construct a theory that combines quantum physics and gravity and for this reason, the investigation of the behaviour of relativistic particles in this context is of considerable interest.

It has been known that the energy levels of an atom placed in a gravitational field will be shifted as a result of the interaction of the atom with spacetime curvature[4-8]. These shifts in the energy levels, which would depend on the features of the spacetime, are different for each energy level, and thus are distinguishable from the Doppler effect and from the gravitational and cosmological redshifts, in which cases these shifts would be the same for all spectral lines. In fact, it was already shown that in the Schwarzschild geometry, the shift in the energy level due to gravitational effects is different from the Stark and Zeeman effects, and therefore, it would be possible, in principle, to separate the shifts in the energy levels caused by electromagnetic and by gravitational perturbations [7]. Thus, in these situations the energy spectrum carries unambiguous information about the local features of the background spacetime in which the atomic system is located.

The general theory of relativity, as a metric theory, predict that gravitation is manifested as curvature of spacetime. This curvature is characterized by the Riemann tensor. On the other hand, we know that there are connections between topological properties of the space and local physical laws in such a way that the local intrinsic geometry of the space is not sufficient to describe completely the physics of a given system. As an example of a gravitational effect of topological origin, we can mention the fact that only when a particle is transported around a cosmic string along a closed curve the string is noticed at all. This situation corresponds to the gravitational analogue [9] of the electromagnetic Aharonov-Bohm effect [10], in which electrons are beamed past a solenoid containing a magnetic field. These effects are of topological origin rather than local. In fact, the nontrivial topology of spacetime, as well as its curvature, leads to a number of interesting gravitational effects. Thus, it is also important to investigate the role played by a nontrivial topology, for example, on a quantum system. As examples of these investigations we can mention the study of the topological scattering in the context of quantum mechanics on a cone [11], and the investigations on the interaction of a quantum system with conical singularities [12,13] and on quantum mechanics on topological defects spacetimes [14].

Therefore, taking into account that we have to consider the topology of spacetime in order to describe completely a given physical system, we want to address the question of how the nontrivial topology could affect the energy levels by shifting the atomic spectral lines. For the purpose of investigating this problem, a calculation of the energy levels shifts of the hydrogen atom is carried out in the spacetimes of an infinitely thin cosmic string [15] and of a point-like global monopole [16].

Topological defects may arise in gauge models with spontaneous symmetry breaking. They can be of various types such as monopoles, domain walls, cosmic strings and their hybrids [17]. They may have been formed during universe expansion and their nature depends on the topology of the vacuum manifold of the theory under consideration [18]. The richness of the new ideas they brought along to general relativity seems to justify the interest in the study of these structures, and specifically the role played by their topological features at the atomic level.

The gravitational field of a cosmic string is quite remarkable; a particle placed at rest around a straight, infinite, static cosmic string will not be attracted to it; there is no local gravity. The spacetime around a cosmic string is locally flat but not globally. The external gravitational field due to a cosmic string may be approximately described by a commonly called conical geometry. The nontrivial topology of this spacetime leads to a number of interesting effects like, for example, gravitational lensing [19], emission of radiation by a freely moving particle [20], electrostatic self-force [21] on an electric charge at rest and the so-called gravitational Aharonov-Bohm effect [9] among other.

The spacetime of a point-like global monopole has also some unusual properties. It possesses a deficit solid angle $\Delta = 32\pi^2 G\eta^2$, η being the energy scale of symmetry breaking. Test particles in this spacetime experiences a topological scattering by an angle

$\pi\Delta/2$ irrespective of their velocity and their impact parameter. Also in this case, the nontrivial topology of spacetime, as well as its curvature, which are due to the deficit solid angle leads to a number of interesting effects [22,23] which are not present in flat Minkowski spacetime.

In this paper, we deal with the interesting problem concerning the modifications of the energy levels of a hydrogen atom placed in the gravitational fields of a cosmic string and of a global monopole. In order to investigate this problem further we determine the solutions of the corresponding Dirac equations and the energy levels of a hydrogen atom under the influence of these gravitational fields. To do these calculations we shall make the following assumptions: (i) The atomic nucleus is not affected by the presence of the defect. (ii) The atomic nucleus is located on the defect. With these, to do our calculations accordingly would have been possible and doing so it affords an explicit demonstration of the effects of spacetime topology on the shifts in the atomic spectral lines of the hydrogen atom.

A similar problem concerning the effects of gravitational fields at atomic level has been considered before. As example of some works on this topic, we can mention [4-8] which obtained the expressions for the shifts in the energy levels of an atom caused by its interaction with the curvature of spacetime and also a recent paper [24] which calculated the atomic energy level shifts of atoms placed in strong gravitational fields near collapsing spheroidal masses.

The results obtained in this paper are related to the previous ones[4-8] connected with this topic, in the sense that we also study the effect of gravitational fields at the atomic level, however, our calculation provides an interesting new example of an effect at atomic scale which can be thought of as a consequence of the nontrivial topology of spacetime and this aspect was not taken into account by previous works[4,8].

In the case of an infinitely thin cosmic string spacetime, the shifts in the energy levels depend on the angle deficit and for the global monopole spacetime these shifts depend on the deficit solid angle. In both situations these effects vanish when these angle deficits vanish, as it should be.

This paper is organized as follows. In section II we obtain the solution of the Dirac equation and we calculate the energy shifts experienced by a hydrogen atom placed in the gravitational field of a cosmic string. In section III we also obtain the solutions of the Dirac equation and we calculate the modifications of the spectrum of a hydrogen atom in the gravitational field of a global monopole. Finally, in section IV, we draw some conclusions.

II. Relativistic hydrogen atom in the spacetime of the cosmic string

In what follows we will study the behaviour of a hydrogen atom in the spacetime of a cosmic string. The line element corresponding to the cosmic string spacetime [15] is given, in spherical coordinates, by

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\theta^2 + \alpha^2r^2\sin^2\theta d\phi^2. \tag{1}$$

The parameter $\alpha = 1 - \frac{4G}{c^2}\bar{\mu}$ runs in the interval $(0, 1]$, with $\bar{\mu}$ being the linear mass density of the cosmic string.

Let us consider the generally covariant form of the Dirac equation which is given by

$$\left[i\gamma^{\mu}(x)\left(\partial_{\mu} + \Gamma_{\mu}(x) + i\frac{eA_{\mu}}{\hbar c}\right) - \frac{\mu c}{\hbar}\right]\Psi(x) = 0, \tag{2}$$

where μ is the mass of the particle, A_{μ} is an external electromagnetic potential and $\Gamma_{\mu}(x)$ are the spinor affine connections which can be expressed in terms of the set of tetrad fields $e^{\mu}_{(a)}(x)$ and the standard flat spacetime $\gamma^{(a)}$ Dirac matrices as

$$\Gamma_{\mu} = \frac{1}{4}\gamma^{(a)}\gamma^{(b)}e^{\nu}_{(a)}(\partial_{\mu}e_{(b)\nu} - \Gamma^{\sigma}_{\mu\nu}e_{(b)\sigma}). \tag{3}$$

The generalized Dirac matrices $\gamma^{\mu}(x)$ satisfies the anticommutation relations

$$\{\gamma^{\mu}(x), \gamma^{\nu}(x)\} = 2g^{\mu\nu}(x),$$

and are defined by

$$\gamma^{\mu}(x) = e^{\mu}_{(a)}(x)\gamma^{(a)}, \tag{4}$$

where $e^{\mu}_{(a)}(x)$ obeys the relation $\eta^{ab}e^{\mu}_{(a)}(x)e^{\nu}_{(b)}(x) = g^{\mu\nu}$; μ, $\nu = 0, 1, 2, 3$ are tensor indices and a, $b = 0, 1, 2, 3$ are tetrad indices.

In this paper, the following explicit forms of the constant Dirac matrices will be taken

$$\gamma^{(0)} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix};\ \gamma^{(i)} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix};\ i = 1,2,3\,, \tag{5}$$

where σ^i are the usual Pauli matrices.

In order to write the Dirac equation in this spacetime, let us take the tetrads $e^{\mu}_{(a)}(x)$ as

$$e^{\mu}_{(a)}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \frac{\cos\theta\cos\phi}{r} & \frac{\cos\theta\sin\phi}{r} & -\frac{\sin\theta}{r} \\ 0 & -\frac{\sin\phi}{\alpha r\sin\theta} & \frac{\cos\phi}{\alpha r\sin\theta} & 0 \end{pmatrix}. \tag{6}$$

Thus using (6), we obtain the following expressions for the generalized Dirac matrices $\gamma^{\mu}(x)$

$$\begin{aligned} \gamma^0(x) &= \gamma^{(0)}, \\ \gamma^1(x) &= \gamma^{(r)}, \\ \gamma^2(x) &= \frac{\gamma^{(\theta)}}{r}, \\ \gamma^3(x) &= \frac{\gamma^{(\phi)}}{\alpha r\sin\theta}, \end{aligned} \tag{7}$$

where

$$\begin{pmatrix} \gamma^{(r)} \\ \gamma^{(\theta)} \\ \gamma^{(\phi)} \end{pmatrix} = \begin{pmatrix} \cos\phi\sin\theta & \sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \sin\phi\cos\theta & -\sin\phi \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \gamma^{(1)} \\ \gamma^{(2)} \\ \gamma^{(3)} \end{pmatrix}. \tag{8}$$

The covariant Dirac Eq. (2), written in the spacetime of a cosmic string is then given by

$$\begin{aligned} &\left[i\hbar \textstyle\sum^r \partial_r + i\hbar \frac{\sum^\theta}{r}\partial_\theta + i\hbar\frac{\sum^\phi}{\alpha r\sin\theta}\partial_\phi \right. \\ &\left. +i\hbar\frac{1}{2r}\left(1-\frac{1}{\alpha}\right)\left(\textstyle\sum^r + \cot\theta \sum^\theta\right) - \frac{eA_0}{c} - \gamma^{(0)}\mu c + \frac{E}{c}\right]\chi(\vec{r}) = 0, \end{aligned} \tag{9}$$

where $\sum^r$, $\sum^\theta$ and $\sum^\phi$ are defined by

$$\textstyle\sum^r \equiv \gamma^{(0)}\gamma^{(r)};\ \sum^\theta \equiv \gamma^{(0)}\gamma^{(\theta)};\ \sum^\phi \equiv \gamma^{(0)}\gamma^{(\phi)}, \tag{10}$$

and we have chosen $\Psi(x)$ as

$$\Psi\left(x\right)=e^{-i\frac{E}{\hbar}t}\chi\left(\vec{r}\right), \tag{11}$$

which comes from the fact that the spacetime under consideration is static.

We must now turn our attention to the solution of the equation for $\chi(\vec{r})$. Then, let us assume that the solutions of Eq. (9) are of the form

$$\chi\left(\vec{r}\right)=r^{-\frac{1}{2}\left(1-\frac{1}{\alpha}\right)}\left(\sin\theta\right)^{-\frac{1}{2}\left(1-\frac{1}{\alpha}\right)}R\left(r\right)\Theta\left(\theta\right)\Phi\left(\phi\right). \tag{12}$$

Thus, substituting Eq. (12) into (9), we obtain the following radial equation

$$\left(c\sum_r' p_r+i\hbar c\frac{\sum_r'}{r}\gamma^{(0)}k_{(\alpha)}+eA_0+\mu c^2\gamma^{(0)}\right)R\left(r\right)=ER\left(r\right). \tag{13}$$

where

$$k_{(\alpha)}=\pm\left(j_{(\alpha)}+\frac{1}{2}\right)=\pm\left[j+\frac{1}{2}+m\left(\frac{1}{\alpha}-1\right)\right] \tag{14}$$

are the eigenvalues of the generalized spin-orbit operator $K_{(\alpha)}$ in the spacetime of a cosmic string and $j_{(\alpha)}$ corresponds to the eigenvalues of the generalized total angular momentum operator. The operator K_α is given by

$$\hbar\gamma^{(0)}K_{(\alpha)}=\vec{\Sigma}\cdot\vec{L}_{(\alpha)}+\hbar, \tag{15}$$

with $\vec{\Sigma}=\left(\Sigma^r,\ \Sigma^\theta,\ \Sigma^\phi\right)$ and $\vec{L}_{(\alpha)}$ is the generalized angular momentum operator [13] in the spacetime of the cosmic string, which is such that $\vec{L}^2_{(\alpha)}Y^{m_{(\alpha)}}_{l_{(\alpha)}}\left(\theta,\phi\right)=\hbar^2 l_{(\alpha)}\left(l_{(\alpha)}+1\right)$, with $Y^{m_{(\alpha)}}_{l_{(\alpha)}}\left(\theta,\phi\right)$ being the generalized spherical harmonics in the sense that $m_{(\alpha)}$ and $l_{(\alpha)}$ are not necessarily integers. The parameters $m_{(\alpha)}$ and $l_{(\alpha)}$ are given, respectively, by $m_{(\alpha)}\equiv\frac{m}{\alpha}$ and $l_{(\alpha)}\equiv n+m_{(\alpha)}=l+m\left(\frac{1}{\alpha}-1\right)$, $l=0,1,2,...\ n-1$, l is the orbital angular momentum quantum number, m is the magnetic quantum number and n is the principal quantum number.

Let us choose the following two-dimensional representation for $\sum_r'$ and $\gamma^{(0)}$

$$\sum_r'\equiv\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix};\ \gamma^{(0)}\equiv\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{16}$$

Now, let us assume that the radial solution can be written as

$$R(r)=\frac{1}{r}\begin{pmatrix}-iF\left(r\right)\\ G\left(r\right)\end{pmatrix}. \tag{17}$$

Then, Eq. (13) decomposes into the coupled equations

$$-i\left(\hbar c\right)^{-1}\left[E-E_0+\frac{e^2}{r}\right]F(r)+\frac{dG(r)}{dr}+\frac{k_{(\alpha)}}{r}G(r)=0, \tag{18}$$

and

$$-i\left(\hbar c\right)^{-1}\left[E+E_0+\frac{e^2}{r}\right]G(r)+\frac{dF(r)}{dr}-\frac{k_{(\alpha)}}{r}F(r)=0, \tag{19}$$

where $E_0=\mu c^2$ is the rest energy of the electron. Note that in obtaining these equations use was made of the fact that $A_0=-e/r$.

The solutions of these equations are given in terms of the confluent hypergeometric function $M(a,b;x)$ as

$$\begin{aligned}F(r)=-i\sqrt{\frac{Q}{T}}\frac{e^{-rD}}{2}\left(rD\right)^{\gamma_{(\alpha)}-1}&\left[M\left(\gamma_{(\alpha)}-1+\tilde{P},2\gamma_{(\alpha)}-1;2rD\right)\right.\\&\left.+\frac{\left(\gamma_{(\alpha)}-1+\tilde{P}\right)}{\left(k_{(\alpha)}+\tilde{Q}\right)}M\left(\gamma_{(\alpha)}+\tilde{P},2\gamma_{(\alpha)}-1;2rD\right)\right],\end{aligned} \tag{20}$$

and

$$\begin{aligned}G(r)=\frac{e^{-rD}}{2}\left(rD\right)^{\gamma_{(\alpha)}-1}&\left[M\left(\gamma_{(\alpha)}-1+\tilde{P},2\gamma_{(\alpha)}-1;2rD\right)\right.\\&\left.-\frac{\left(\gamma_{(\alpha)}-1+\tilde{P}\right)}{\left(k_{(\alpha)}+\tilde{Q}\right)}M\left(\gamma_{(\alpha)}+\tilde{P},2\gamma_{(\alpha)}-1;2rD\right)\right],\end{aligned} \tag{21}$$

where $T=\frac{E_0-E}{\hbar c}$; $Q=\frac{E_0+E}{\hbar c}$, $D=\sqrt{TQ}=\frac{\sqrt{E_0^2-E^2}}{\hbar c}$; $\gamma_{(\alpha)}=1+\sqrt{k_{(\alpha)}^2-\tilde{\alpha}^2}$; $\tilde{P}\equiv\frac{\tilde{\alpha}}{2}\left(\sqrt{T/Q}-\sqrt{Q/T}\right)$; $\tilde{Q}\equiv\frac{\tilde{\alpha}}{2}\left(\sqrt{T/Q}+\sqrt{Q/T}\right)$, with $\tilde{\alpha}=\frac{e^2}{\hbar c}\approx\frac{1}{137}$ being the fine structure constant.

The solutions given by (20) and (21) are divergent, unless the following condition is fulfilled

$$\gamma_{(\alpha)}-1+\tilde{P}=-n;\ n=0,1,2..., \tag{22}$$

which means that

$$\frac{1}{2}\tilde{\alpha}\left(\sqrt{\frac{T}{Q}}-\sqrt{\frac{Q}{T}}\right)=-\left(n+\gamma_{(\alpha)}-1\right). \tag{23}$$

From this equation we may infer that the energy eigenvalues are given by

$$E=E_0\left[1+\tilde{\alpha}^2\left(n+\left|k_{(\alpha)}\right|\sqrt{1-\tilde{\alpha}^2k_{(\alpha)}^{-2}}\right)^{-2}\right]^{-\frac{1}{2}}. \tag{24}$$

This equation exhibits the angle deficit dependence of the energy levels. It is helpful to introduce the quantum number $n_{(\alpha)}$ that corresponds to the principal quantum number of the nonrelativistic theory when $\alpha = 1$,

$$n_{(\alpha)} = n + j_{(\alpha)} + \frac{1}{2}. \tag{25}$$

Therefore, Eq. (24) may be cast in the form

$$E_{n_{(\alpha)},j_{(\alpha)}} = E_0 \left\{ 1 + \tilde{\alpha}^2 \left[\left(n_{(\alpha)} - j_{(\alpha)} - \frac{1}{2} \right) + \left(j_{(\alpha)} + \frac{1}{2} \right) \sqrt{1 - \tilde{\alpha}^2 \left(j_{(\alpha)} + \frac{1}{2} \right)^{-2}} \right]^{-2} \right\}^{-\frac{1}{2}}. \tag{26}$$

This equation can be written in a way which is better suited to physical interpretation. Thus, as $\tilde{\alpha} \ll 1$, we can expand Eq. (26) in a powers of $\tilde{\alpha}$, and as a result we get the following leading terms

$$E_{n_{(\alpha)},j_{(\alpha)}} = E_0 - E_0 \frac{\tilde{\alpha}^2}{2n_{(\alpha)}^2} + E_0 \frac{\tilde{\alpha}^4}{2n_{(\alpha)}^4} \left(\frac{3}{4} - \frac{n_{(\alpha)}}{j_{(\alpha)} + \frac{1}{2}} \right). \tag{27}$$

The first term corresponds to the rest energy of the electron; the second one gives the energy of the bound states in the non-relativistic approximation and the third one corresponds to the relativistic correction. Note that these last two terms depend on the deficit angle. The further terms can be neglected in comparison with these first three terms.

Now, let us consider the total shift in the energy between the states with $j = n - \frac{1}{2}$, and $j = \frac{1}{2}$, for a given n. This shift is given by

$$\Delta E_{n_{(\alpha)},j_{(\alpha)}} = \frac{\mu e^8}{\hbar^4 c^2 n_{(\alpha)}^3} \left(\frac{n_{(\alpha)} - 1}{2 \left[n_{(\alpha)} + m \left(\frac{1}{\alpha} - 1 \right) \right] \left[1 + m \left(\frac{1}{\alpha} - 1 \right) \right]} \right). \tag{28}$$

One important characteristic of Eq. (26) is that it contains a dependence on n, j and α. The dependence on α corresponds to an analogue of the electromagnetic Aharonov-Bohm effect for bound states, but now in the gravitational context. Therefore, the interaction with the topology(conical singularity) causes the energy levels to change. Note that the presence of the cosmic string destroys the degeneracy of all the levels, corresponding to $l = 0$ and $l = 1$, and destroys partially this degeneracy for the other sublevels. Therefore, as the occurrence of degeneracy can often be ascribed to some symmetry property of the physical system, the fact that the presence of the cosmic string destroys the degeneracy means that there is a break of the original symmetry. Observe that for $\alpha = 1$, the results reduce to the flat Minkowski spacetime case as expected.

As a estimation of the effect of the cosmic string on the energy shift of the hydrogen atom, let us consider $\alpha = 1 - 10^{-6}$ which corresponds to GUT cosmic strings. Using this value into Eq. (43), we conclude that the presence of the cosmic string reduces the energy of the level of the states $2P_{1/2}(n = 2,\ l = 1,\ j = l - \frac{1}{2} = \frac{1}{2},\ m = 1)$ to about $10^{-4}\%$ in comparison with the flat spacetime value. This decrease is of the order of the measurable Zeeman effect in carbon atoms for $2P$ states when submitted, for example, to an external magnetic field with strength to about tens of Tesla. Therefore, this shift in energy levels produced by a cosmic string is measurable as well.

Finally, we can write down the general solution to Eq. (2) corresponding to a hydrogen atom placed in the background spacetime of a cosmic string. Thus, it reads

$$\begin{aligned} \Psi_{l_{(\alpha)},j_{(\alpha)}=l_{(\alpha)}+\frac{1}{2},m_{(\alpha)}}(x) &= e^{-i\frac{Et}{\hbar}} r^{-\frac{1}{2}\left(1-\frac{1}{\alpha}\right)} (\sin\theta)^{-\frac{1}{2}\left(1-\frac{1}{\alpha}\right)} \\ &\quad\times F_{(\alpha)}(r) \begin{pmatrix} \sqrt{\frac{l_{(\alpha)}+m_{(\alpha)}+\frac{1}{2}}{2l_{(\alpha)}+1}}\, Y_{l(\alpha)}^{m_{(\alpha)}-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{l_{(\alpha)}-m_{(\alpha)}+\frac{1}{2}}{2l_{(\alpha)}+1}}\, Y_{l(\alpha)}^{m_{(\alpha)}+\frac{1}{2}}(\theta,\phi) \end{pmatrix}, \end{aligned} \tag{29}$$

and

$$\begin{aligned} \Psi_{l_{(\alpha)},j_{(\alpha)}=l_{(\alpha)}-\frac{1}{2},m_{(\alpha)}}(x) &= e^{-i\frac{Et}{\hbar}} r^{-\frac{1}{2}\left(1-\frac{1}{\alpha}\right)} (\sin\theta)^{-\frac{1}{2}\left(1-\frac{1}{\alpha}\right)} \\ &\quad\times G_{(\alpha)}(r) \begin{pmatrix} -\sqrt{\frac{l_{(\alpha)}-m_{(\alpha)}+\frac{1}{2}}{2l_{(\alpha)}+1}}\, Y_{l(\alpha)}^{m_{(\alpha)}-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{l_{(\alpha)}+m_{(\alpha)}+\frac{1}{2}}{2l_{(\alpha)}+1}}\, Y_{l(\alpha)}^{m_{(\alpha)}+\frac{1}{2}}(\theta,\phi) \end{pmatrix}, \end{aligned} \tag{30}$$

where $F_{(\alpha)}(r)$ and $G_{(\alpha)}(r)$ are given by Eqs. (20) and (21), respectively, and the index α was introduced to emphasize the dependence of these functions on this parameter.

Note that the solutions depend on the topological features of the spacetime of a cosmic string whose influence appears codified in the parameter α associated with the presence of the cosmic string and this is the point at issue here.

III. Relativistic hydrogen atom in the presence of a global monopole

In continuation of the preceding consideration, in this section we shall be concerned with the study of the influence of a global monopole on the states of a hydrogen atom.

The solution corresponding to a global monopole in a $O(3)$ broken symmetry model has been investigated by Barriola and Vilenkin [16].

Far away from the global monopole core we can neglect the mass term and as a consequence the main effects are produced by the solid deficit angle. The respective metric in the Einstein theory of gravity can be written as [16]

$$ds^2 = -c^2dt^2 + dr^2 + b^2r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{31}$$

where $b^2 = 1 - 8\pi G\eta^2$, the parameter η being the energy scale of symmetry breaking.

Now, let us choose the tetrad as

$$e^{\mu}_{(a)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \frac{\cos\theta\cos\phi}{br} & \frac{\cos\theta\sin\phi}{br} & -\frac{\sin\theta}{br} \\ 0 & -\frac{\sin\phi}{br\sin\theta} & \frac{\cos\phi}{br\sin\theta} & 0 \end{pmatrix}. \tag{32}$$

Therefore, the generalized and flat spacetime Dirac matrices are related by

$$\begin{aligned} \gamma^0(x) &= \gamma^{(0)}, \\ \gamma^1(x) &= \gamma^{(r)}, \\ \gamma^2(x) &= \frac{\gamma^{(\theta)}}{br}, \\ \gamma^3(x) &= \frac{\gamma^{(\phi)}}{br\sin\theta}. \end{aligned} \tag{33}$$

where, $\gamma^{(r)}$, $\gamma^{(\theta)}$and $\gamma^{(\phi)}$ were defined in the previous section.

Proceeding in analogy with section II we find that the generalized Dirac equation can be written, in this background spacetime, as

$$\begin{aligned} &\left[i\hbar\sum^r\partial_r + i\hbar\frac{\sum^\theta}{br}\partial_\theta + i\hbar\frac{\sum^\phi}{br\sin\theta}\partial_\phi \right. \\ &\left. + i\hbar\frac{1}{r}\left(1-\frac{1}{b}\right)\left(\sum^r + \cot\theta\sum^\theta\right) + \frac{e^2}{rc} - \gamma^{(0)}\mu c + \frac{E}{c}\right]\chi(\vec{r}) = 0\,, \end{aligned} \tag{34}$$

where Eq. (11) has been used in obtaining the above result.

Now, let us assume that the solution of Eq. (34) can be written as

$$\chi\left(\vec{r}\right)=r^{-\left(1-\frac{1}{b}\right)}R\left(r\right)\Theta\left(\theta\right)\Phi\left(\phi\right), \tag{35}$$

Using Eq. (35), Eq. (34) turns into the simple form

$$\left(c\sum\nolimits_{r}' p_{r}+i\hbar c\frac{\sum_{r}'}{r}\gamma^{(0)}k_{(b)}-\frac{e^{2}}{r}+\mu c^{2}\gamma^{(0)}\right)R\left(r\right)=ER\left(r\right), \tag{36}$$

where

$$\begin{aligned} k_{(b)} &= \pm\left(\frac{j^{2}}{b^{2}}+\frac{j}{b^{2}}+\frac{1}{4}\right)^{\frac{1}{2}} \\ &= \pm\left[\left(\frac{j}{b}+\frac{1}{2}\right)^{2}+\frac{j}{b}\left(\frac{1}{b}-1\right)\right]^{\frac{1}{2}} \end{aligned} \tag{37}$$

are the eigenvalues of the generalized spin-orbit operator $K_{(b)}$ in the spacetime of a global monopole which is given by

$$\hbar K_{(b)}=\gamma^{(0)}\left[\vec{\sum}''\cdot\vec{L}_{(b)}+\hbar\right],$$

where $\sum'' = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$.

In the present case the generalized angular momentum will be denoted by $L_{(b)}$. It is such that $\vec{L}^{2}_{(b)}Y_{l}^{m}\left(\theta,\phi\right)=\frac{\hbar^{2}}{b^{2}}l\left(l+1\right),\ l=0,1,2,...\ n-1$, and $\vec{L}_{(b)}=\frac{\vec{L}}{b}$ is the angular momentum in the spacetime of a global monopole [22]. Using the same procedure as in the previous section, we find

$$\begin{aligned} F_{(b)}(r) = -i\sqrt{\frac{Q}{T}}\frac{e^{-rD}}{2}\left(rD\right)^{\gamma_{(b)}-1}\Big[M\left(\gamma_{(b)}-1+\tilde{P},2\gamma_{(b)}-1;2rD\right) \\ +\frac{\left(\gamma_{(b)}-1+\tilde{P}\right)}{\left(k_{(b)}+\tilde{Q}\right)}M\left(\gamma_{(b)}+\tilde{P},2\gamma_{(b)}-1;2rD\right)\Big], \end{aligned} \tag{38}$$

and

$$\begin{aligned} G_{(b)}(r) = \frac{e^{-rD}}{2}\left(rD\right)^{\gamma_{(b)}-1}\Big[M\left(\gamma_{(b)}-1+\tilde{P},2\gamma_{(b)}-1;2rD\right) \\ -\frac{\left(\gamma_{(b)}-1+\tilde{P}\right)}{\left(k_{(b)}+\tilde{Q}\right)}M\left(\gamma_{(b)}+\tilde{P},2\gamma_{(b)}-1;2rD\right)\Big], \end{aligned} \tag{39}$$

where $\gamma_{(b)}=1+\sqrt{k^{2}_{(b)}-\tilde{\alpha}^{2}}$; T, Q, M, $\tilde{P}$ and $\tilde{Q}$ are the same defined previously. The index b in the functions F and G indicates their dependence on this parameter. These functions are, formally, the same used in the previous section.

By the use of condition (22) with the interchange of $\gamma_{(\alpha)}$ by $\gamma_{(b)}$, we obtain the following spectrum of energy eigenvalues

$$E_{n_{(b)},j_{(b)}} = E_0 \left\{ 1 + \tilde{\alpha}^2 \left[n_{(b)} - \left| k_{(b)} \right| + \left| k_{(b)} \right| \sqrt{1 - \tilde{\alpha}^2 k_{(b)}^{-2}} \right]^{-2} \right\}^{-\frac{1}{2}} . \tag{40}$$

in which we have defined $n_{(b)}$ as a number which reduces to the principal quantum number when $b = 1$ and is given by

$$n_{(b)} = n + \left| k_{(b)} \right| = n + \left[\left(\frac{j}{b} + \frac{1}{2} \right)^2 + \frac{j}{b} \left(\frac{1}{b} - 1 \right) \right]^{\frac{1}{2}} . \tag{41}$$

Then, expanding Eq. (40) in a series of powers of $\tilde{\alpha}$, we have the following leading terms

$$E_{n_{(b)},j_{(b)}} = E_0 - E_0 \frac{\tilde{\alpha}^2}{2n_{(b)}^2} + E_0 \frac{\tilde{\alpha}^4}{2n_{(b)}^4} \left(\frac{3}{4} - \frac{n_{(b)}}{\left[\left(\frac{j}{b} + \frac{1}{2} \right)^2 + \frac{j}{b} \left(\frac{1}{b} - 1 \right) \right]^{\frac{1}{2}}} \right), \tag{42}$$

which tell us what the dependence of each term with the parameter b is. In this case, the shift in the energy between the energy levels with $j = n - \frac{1}{2}$ and $j = \frac{1}{2}$, for a given n, is

$$\Delta E_{n_{(b)},j_{(b)}} = \frac{\mu e^8}{2\hbar^4 c^2 n^3} \left\{ \frac{\left[\left(\frac{n_{(b)}}{b} - \frac{1}{2} \left(\frac{1}{b} - 1 \right) \right)^2 + \left(\frac{n_{(b)}}{b} - \frac{1}{2b} \right) \left(\frac{1}{b} - 1 \right) \right]^{\frac{1}{2}} - \left[\left(\frac{1}{2b} + \frac{1}{2} \right)^2 + \frac{1}{2b} \left(\frac{1}{b} - 1 \right) \right]^{\frac{1}{2}}}{\left[\left(\frac{1}{2b} + \frac{1}{2} \right)^2 + \frac{1}{2b} \left(\frac{1}{b} - 1 \right) \right]^{\frac{1}{2}} \left[\left(\frac{n_{(b)}}{b} - \frac{1}{2} \left(\frac{1}{b} - 1 \right) \right)^2 + \left(\frac{n_{(b)}}{b} - \frac{1}{2b} \right) \left(\frac{1}{b} - 1 \right) \right]^{\frac{1}{2}}} \right\} . \tag{43}$$

This equation reduces to the same result of the flat spacetime in the absence of the global monopole ($b = 1$).

It is worth noticing from Eq. (42) that the presence of the monopole does not break the degeneracy of the energy levels and as in the case of a cosmic string.

As an estimation of the shift in the energy levels, let us consider a grand unified (GUT) monopole in which $b^2 = 1 - 10^{-6}$. Using this value into Eq. (42) we conclude that the presence of the monopole reduces the relativistic correction of the energy of the level $2P_{1/2}(n = 2,\ l = 1,\ j = l - \frac{1}{2} = \frac{1}{2},\ m = 1)$ in approximately $10^{-4}\%$ as compared with the result of the flat Minkowski spacetime.

Finally, let us write down the general solution for this case. It reads as

$$\Psi_{l,j=l+\frac{1}{2},m}\left(x\right) = e^{-i\frac{Et}{\hbar}}r^{-\left(1-\frac{1}{b}\right)} \times F_{(b)}\left(r\right)\begin{pmatrix} \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}Y_l^{m-\frac{1}{2}}\left(\theta,\phi\right) \\ \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}Y_l^{m+\frac{1}{2}}\left(\theta,\phi\right) \end{pmatrix}, \tag{44}$$

and

$$\Psi_{l,j=l-\frac{1}{2},m}\left(x\right) = e^{-i\frac{Et}{\hbar}}r^{-\frac{1}{2}\left(1-\frac{1}{b}\right)} \times G_{(b)}\left(r\right)\begin{pmatrix} -\sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}Y_l^{m-\frac{1}{2}}\left(\theta,\phi\right) \\ \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}Y_l^{m+\frac{1}{2}}\left(\theta,\phi\right) \end{pmatrix}, \tag{45}$$

where $F_{(b)}\left(r\right)$ and $G_{(b)}\left(r\right)$ are given by Eqs. (38) and (39), respectively. It is important to call attention to the fact that all these results depends on the geometrical and topological features of the global monopole spacetime.

IV. CONCLUSIONS

With the purpose of discussing the role of the topology on an atomic system we carried out the calculations of the shifts in the energy levels of hydrogen atom placed in the spacetimes of a string and a monopole, adding, in this way, some new results to the interesting problem considered in seminal papers by Parker and collaborators[4-8] about the effects of gravitational fields at the atomic level, but now from the geometrical and topological points of view, instead of looking only for the local effects of the curvature as in those earlier papers[4-8].

The presence of a cosmic string changes the solution and shifts the energy levels of a hydrogen atom as compared with the flat Minkowski spacetime result. It is interesting to observe that these shifts depend on the parameter that defines the angle deficit and vanish when the angle deficit vanishes. These shifts arise from the topological features of the spacetime generated by this defect.

In the case of the hydrogen atom in the spacetime of a global monopole, the modifications in the solution and the shifts in the energy levels are due to the combined effects of the curvature and the nontrivial topology determined by the deficit solid angle associated with this spacetime. These shifts also vanish when the deficit solid angle vanishes.

Both effects can be thought of as a consequence of the topological influence of the spacetime under consideration upon the hydrogen atom.

The decrease in the energy for the situations considered is only two orders of magnitude less than the ratio between the fine structure splitting and the energy of the ground state of the non-relativistic hydrogen atom and is of the order of the Zeeman effect. Therefore, the modifications in the spectra of the hydrogen atom due to the presence of the gravitational fields of a string or a monopole are all measurable, in principle.

The obtained results show how the geometry and a nontrivial topology influences the energy spectrum as compared with the flat spacetime case and show how these quantities depend on the surroundings and their characteristics. These results also show how the solutions are modified.

Therefore, the problem of finding how the energy spectrum of an atom placed in a gravitational field is perturbed by this background has to take into account not only the geometrical, but also the topological features of the spacetimes under consideration. In other words, the behaviour of an atomic system is determined not only by the curvature at the position of the atom, but also by the topology of the background spacetime.

ACKNOWLEDGMENTS

We acknowledge Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES-Program PROCAD) for partial financial support.

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